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A new interior solution of Einstein's field equations for a spherically symmetric perfect fluid in shear-free motion

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Abstract. Although spherically symmetric expanding (or contracting) stars are a very important class of objects for the application of general relativity theory, only very few perfect fluid solutions of Einstein's equations have been found so far. In this paper, a new class of exact solutions is presented which contains two arbitrary functions of time and one additional parameter.

1. Line element and field equations

It is well known that for shear-free motion of a perfect fluid a coordinate system can be introduced which is simultaneously comoving and isotropic:

$$\begin{aligned} ds^2 &= \exp[2\lambda(r,t)][dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)] - \exp[2\nu(r,t)]\, dt^2, \\ u_i &= (0, 0, 0, -e^{\nu}). \end{aligned} \tag{1.1}$$

If, moreover, the motion has non-vanishing expansion ($u^a_{;a} \neq 0 \Rightarrow \partial\lambda/\partial t \neq 0$), then with

$$L(x,t) \equiv \exp[-\lambda(r,t)], \qquad x \equiv r^2, \tag{1.2}$$

the field equations can be reduced to a single ordinary differential equation

$$L_{,xx} = L^2 F(x), \tag{1.3}$$

the metric function e^{ν} being given by

$$e^{\nu} = \lambda_{,t}\, e^{-f(t)}. \tag{1.4}$$

To obtain a metric, one has to specify $f(t)$ and $F(r^2)$ and find a solution of (1.3). The rest mass density μ and pressure p of the fluid can then be computed from e.g.

$$\begin{aligned} \varkappa_0\mu &= 3\, e^{2f} - e^{-2\lambda}(2\lambda_{,rr} + \lambda_{,r}^2 + 4\lambda_{,r}/r), \\ \varkappa_0 p\lambda_{,t} &= e^{-3\lambda}\, \partial_t[e^{\lambda}(\lambda_{,r}^2 + 2\lambda_{,r}/r) - \exp(3\lambda + 2f)] \end{aligned} \tag{1.5}$$

(see Kramer *et al* (1980) for further details and references).

To find an exact solution for a spherically symmetric perfect fluid in shear-free (and expanding or contracting) motion means to find a solution $L(x,t)$ of (1.3), the function $F(x)$ being suitably chosen. Since (because of $\lambda_{,t} \neq 0$) the function L must depend on t, whereas F does not, the task is more difficult than it may look at the first glance. In practice, the problem should be formulated as follows: find a function

$F(x)$, for which a solution $y(x; \varphi_0, \psi_0)$ of

$$y'' = y^2 F(x) \tag{1.6}$$

can be constructed which contains two (or one) arbitrary constants φ_0, ψ_0. In the context of equation (1.3), these constants are arbitrary functions of time which ensure $L_{,t} \neq 0$.

2. How to find suitable functions $F(x)$

The only systematic way to check whether a given ordinary differential equation of, say, second order can be solved by means of quadratures seems to be that of Lie, treated *in extenso* in his textbook (Lie 1912). The main idea is to ask whether the differential equation is invariant when performing an infinitesimal transformation

$$\begin{aligned} U &= \xi(x, y)\partial_x + \eta(x, y)\partial_y + \eta'\partial y', \\ \eta' &\equiv \partial\eta/\partial x + y'(\partial\eta/\partial y - \partial\xi/\partial x) - y'^2\partial\xi/\partial y. \end{aligned} \tag{2.1}$$

If two or more such transformations U exist, then the differential equation can be solved by quadratures and elimination procedures. If only one transformation U exists, then a reduction to a first-order differential equation is possible.

When applying this scheme to the differential equation (1.6) it turns out that an invariance transformation U will exist only if $F(x)$ is a solution of the fourth-order differential equation

$$\mathrm{d}^3B(x)/\mathrm{d}x^3 \equiv (\mathrm{d}^3/\mathrm{d}x^3)\left(aF^{-2/5} - \tfrac{2}{5}cF^{-2/5}\int F^{2/5}\,\mathrm{d}x\right) = 4(ex+g)F, \tag{2.2}$$

the constants a, c, e and g being suitably chosen. If (2.2) is satisfied, then U can be constructed by means of

$$\xi = B(x), \qquad \eta = (\tfrac{1}{2}B' + c)y + ex + g. \tag{2.3}$$

Unfortunately, a full and detailed explanation of Lie's method would need too much space, so that we have to refer to his textbook. Here, we shall simply give a list of solutions $F(x)$ of equation (2.2) for which the differential equation $y'' = F(x)y^2$ can either be reduced to a first-order differential equation (§ 3) or can be solved completely by means of quadratures (§ 4). Of course, the reader can understand and check all calculations without having read Lie's textbook.

Since the line element (1.1) is invariant under the substitution $\hat{r} = r^{-1}$, functions F and $\hat{F}$ which are connected by $\hat{F}(x) = x^{-5}F(x^{-1})$ will give the same metric and need not be considered separately. In this sense, e.g. $F = x^{-15/7}$ and $F = x^{-20/7}$ are equivalent. On the other hand, functions $F(x)$ and $\tilde{F} = aF(x+b)$ will give different metrics, although the corresponding solutions of (1.6) are rather trivially related to each other.

Obviously Kustaanheimo and Qvist (1948) were the first to apply Lie's method to the problem of spherically symmetric solutions, but they did not find all the solutions of (2.2).

3. Functions $F(x)$ for which a first integral of $y''=y^2F(x)$ is known

In this section, we have (in the form of a list) collected those functions $F(x)$ for which $y''=y^2F(x)$ can be reduced to a first-order differential equation, together with these differential equations. In general, these are Abelian differential equations, and their solutions cannot be given in a closed form. The subcases where $y''=y^2F(x)$ can be solved completely are treated in § 4.

Only in the first example (3.1) will the procedure be explained in some detail. The other cases are treated along the same lines and the same notation will be used.

3.1. $F=x^n$

With $F=x^n$, also $F=a(x+b)^n$ is covered. If one makes the ansatz

$$y'=x^{-(n+3)}f(yx^{n+2}), \tag{3.1a}$$

then the function $f=f(u)$ has to satisfy

$$\mathrm{d}f/\mathrm{d}u=[(n+3)f+u^2]/[(n+2)u+f]. \tag{3.1b}$$

If a solution $f=f(u,\varphi_0)$ is known, then

$$\psi_0=-\ln x+\int\frac{\mathrm{d}u}{(n+2)u+f(u,\varphi_0)},\qquad u=yx^{n+2}, \tag{3.1c}$$

gives an implicit representation of the general solution $y=y(x,\varphi_0,\psi_0)$ of equation (1.6). In the context of equation (1.3), φ_0 and ψ_0 are arbitrary functions of time.

3.2. $F=\mathrm{e}^x$

$$y'=yf(y\,\mathrm{e}^x), \tag{3.2a}$$

$$\mathrm{d}f/\mathrm{d}u=(u-f^2)/u(1+f), \tag{3.2b}$$

$$\psi_0=-x+\int\frac{\mathrm{d}u}{u[1+f(u,\varphi_0)]},\qquad u=y\,\mathrm{e}^x. \tag{3.2c}$$

3.3. $F=(x+\alpha)^n(x+\beta)^{-n-5}$

$$y'=[y/(x+\alpha)(x+\beta)]\{x+f[y(x+\alpha)^{n+2}(x+\beta)^{-(n+3)}]\}, \tag{3.3a}$$

$$\mathrm{d}f/\mathrm{d}u=\{u-[f-\tfrac{1}{2}(\alpha+\beta)]^2+[\tfrac{1}{2}(\alpha-\beta)]^2\}/\{u[f-\tfrac{1}{2}(\alpha+\beta)-(2n+5)\tfrac{1}{2}(\alpha-\beta)]\}, \tag{3.3b}$$

$$\begin{aligned}\psi_0&=\int\frac{\mathrm{d}x}{(x+\alpha)(x+\beta)}=\int\frac{\mathrm{d}u}{u[f(u,\varphi_0)-\frac{1}{2}(\alpha+\beta)-(2n+5)\frac{1}{2}(\alpha-\beta)]},\\ u&=y(x+\alpha)^{n+2}(x+\beta)^{-(n+3)}.\end{aligned} \tag{3.3c}$$

4. Functions $F(x)$ for which $y''=y^2F(x)$ can be completely integrated

The explicit solution $y(x;\varphi_0,\psi_0)$ of $y''=y^2F(x)$ is known only in a few subcases of the functions $F(x)$ given in § 3.

4.1. $F(x)=(ax^2+2bx+c)^{-5/2}$

This class covers the subcases $n=-\frac{5}{2}$ of (3.3) (for $a\neq 0$), and $n=-\frac{5}{2}$ and $n=0$ of (3.1) (for $a=0$). The corresponding solutions were (in the general case) first found by Kustaanheimo and Qvist (cf also Kramer *et al* 1980). For the sake of completeness, we shall give the relevant formulae

$$\begin{aligned}&\psi_0=\int\frac{\mathrm{d}x}{ax^2+2bx+c}-\int\frac{\mathrm{d}u}{[\frac{2}{3}u^3+(b^2-ac)u^2+\varphi_0]^{1/2}},\\ &u=y(ax^2+2bx+c)^{-1/2}\end{aligned}\qquad \text{for } F\neq 0, \quad (4.1)$$

$$y=\varphi_0 x+\psi_0 \qquad \text{for } F=0. \quad (4.2)$$

4.2. $F(x)=x^{-15/7}$

In this case a first integral can be constructed which reads

$$\begin{aligned}&\varphi_0=\tfrac{1}{4}7^3x^{6/7}y'^2-\tfrac{3}{2}49uy'-6y'-\tfrac{21}{4}x^{-6/7}u^2-\tfrac{1}{6}7^3x^{-6/7}u^3,\\ &u=yx^{-1/7}.\end{aligned}\quad (4.3)$$

Note that this integral has *not* been found by applying equation (3.1*b*). The general solution of $y''=y^2x^{-15/7}$ can be obtained by inserting $y'(x,y;\varphi_0)$—as given by (4.3)—into

$$\psi_0=\int\frac{\mathrm{d}y-y'\,\mathrm{d}x}{\frac{49}{4}x^{-1/7}y+1-\frac{1}{12}7^3x^{6/7}y}. \quad (4.4)$$

This leads to

$$\begin{aligned}&\psi_0=\tfrac{12}{49}x^{1/7}+\frac{12}{7^3}\int\frac{-x^{1/7}\mathrm{d}u+[\frac{2}{7}u+(12/7^3)]x^{-6/7}\mathrm{d}x}{[(4/7^3)\varphi_0x^{6/7}+\frac{2}{3}u^3+\frac{12}{49}u^2+(72/7^4)u+(12^2/7^6)]^{1/2}},\\ &u=yx^{-1/7}.\end{aligned}\quad (4.5)$$

5. Concluding remarks

Of course it is always nice to have a class of exact solutions, but—as often—it is a rather difficult task to extract physical information, since metric, mass density and pressure are given only implicitly. To obtain e.g. the metric function e^{λ}, one should solve (4.5) for $u=u[x,\varphi_0(t),\psi_0(t)]$ to get $e^{-\lambda}=r^{2/7}u[r^2,\varphi_0,\psi_0]$. To discuss its dependence on, say, $\varphi_0(t)$ seems to be an intractable problem, although the integral (4.5) can be expressed in terms of elliptic functions. Even in the simplest case $F=0$ of equation (4.2) the detailed discussion of the solution was a paper in its own right (Bonnor and Faulkes 1967).

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